Fibrations II - The Fundamental Lifting Property

Tyrone Cutler

July 13, 2020

Contents

1 The Fundamental Lifting Property

This section is devoted to stating Theorem 1.1 and beginning its proof. We prove only the first of its two statements here. This part of the theorem will then be used in the sequel, and the proof of the second statement will follow from the results obtained in the next section. We will be careful to avoid circular reasoning.

The utility of the theorem will soon become obvious as we repeatedly use its statement to produce maps having very specific properties. However, the true power of the theorem is not unveiled until § 5, where we show how it leads to a mutual characterisation of cofibrations and fibrations in terms of an *orthogonality* relation.

Theorem 1.1 *Let $j : A \hookrightarrow X$ be a closed cofibration and $p : E \to B$ a fibration. Assume given the solid part of the following strictly commutative diagram*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & E \arrow[d, "p"] \\
X \arrow[r, "g"'] \arrow[ur, dashed, "h"] & B.
\end{tikzcd}
\tag{1.1}
$$

Then the dotted filler can be completed so as to make the whole diagram commute if either of the following two conditions are met

- *j is a homotopy equivalence.*
- *p is a homotopy equivalence.*

As mentioned above, we prove here only the case that j is a homotopy equivalence. The proof of this is given just after the following lemma. The proof of the second statement can be found in § 5 and relies on the material of § 4.

Lemma 1.2 *Let X be a space and assume that there is a function $\varphi : X \to I$ such that the inclusion $i : A = \varphi^{-1}\{0\} \subseteq X$ is a strong deformation retract. Suppose given the solid part of the following commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] \arrow[ur, dashed, "h"] & B. \end{tikzcd} \tag{1.2}$$

If p is a fibration, then the dotted filler $h : X \to E$ may be completed so as to make the whole diagram commute strictly.

Proof Fix a retraction $r : X \to A$ and a homotopy $D : id_X \simeq ir$ under A, and use this data to define $G : X \times I \to X$ by setting

$$G(x,t) = \begin{cases} D(x, \min\{1, t/\varphi(x)\}) & t < \varphi(y) \\ x & t \geq \varphi(x). \end{cases} \tag{1.3}$$

Then, since p is a fibration, there exists a homotopy $\widehat{G}$ completing the following diagram

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "in_0"'] & A \arrow[r, "f"] & E \arrow[d, "p"] \\ X \times I \arrow[r, "G"'] \arrow[urr, dashed, "\widehat{G}"] & X \arrow[r, "g"'] & B. \end{tikzcd} \tag{1.4}$$

Finally we get the map $h : X \to E$ by setting

$$h(x) = \widehat{G}(x, \varphi(x)). \quad \blacksquare \tag{1.5}$$

Proof of 1.1 when j is a homotopy equivalence If $j : A \hookrightarrow X$ is a cofibration and a homotopy equivalence, then we know from the exercises that A is a strong deformation retract of X. If moreover A is closed in X, then there is a map $\varphi : X \to I$ with $\varphi^{-1}(0) = A$ which forms part of a Strøm structure. Thus the conditions for Lemma 1.2 are met and we can find a lift h against the fibration p. $\blacksquare$

We end this section with an important example to give the reader an idea of how Theorem 1.1 may be used effectively.

Lemma 1.3 *If $A \hookrightarrow X$ is a cofibration, then so are the inclusions*

1) $X \times 0 \cup A \times I \hookrightarrow X \times I$

2) $X \times \partial I \cup A \times I \to X \times I$.

Proof We prove only that 1) is a cofibration, since the proof for 2) is similar. Fix a homeomorphism $\alpha : I \times I \to I \times I$ which maps $0 \times I \cup I \times 0$ onto $I \times 0$. Then $id_X \times \alpha : X \times I \times I \to X \times I \times I$ is a homeomorphism which maps $X \times I \times 0 \cup X \times 0 \times I \cup A \times I \times I$ onto $X \times I \times 0 \cup A \times I \times I$. Now, since $A \times I \hookrightarrow X \times I$ is a cofibration we can find a retraction $r : X \times I \times I \to X \times I \times 0 \cup A \times I \times I$. The composite $r(id_X \times \alpha)^{-1}$ is now a retraction

$$X \times I \times I \to X \times I \times 0 \cup (X \times 0 \cup A \times I) \times I. \quad \blacksquare \tag{1.6}$$

We have just seen that the inclusion $X \times 0 \cup A \times I \hookrightarrow X \times I$ is a cofibration when $A \hookrightarrow X$ is. In fact its easy to see that it is also a deformation retraction. In particular, if $p : E \to B$ is a fibration, then according to Theorem 1.1 a filler can be found for any diagram of the form

$$\begin{tikzcd} X \times 0 \cup A \times I \arrow[r, "f \cup G"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B. \end{tikzcd} \tag{1.7}$$

Observe the input data here:

- A map $f : X \to E$.
- A partial homotopy $G : A \times I \to E$ starting at $f|_A$.
- A homotopy $H : X \times I \to B$ starting at pf.

The output is the homotopy $\widetilde{H} : X \times I \to E$ lifiting H and extending G. The existence of $\widetilde{H}$ is referred to as the **covering homotopy extension property** of p. $\square$

2 Spaces Over B

In this section B will denote an arbitrary but fixed space. We will define and study the category of *spaces over* B. Later we will show how to do homotopy theory in this category. Such categories are also called *overcategories*, or *slice categories*, and are the correct settings for parametrised problems which arise frequently in applications of topology and homotopy to other areas of mathematics.

The category of spaces over B is also the correct setting to formulate questions about fibrations with codomain B. It is especially in this context that we will come to understand the true power of fibrations.

Definition 1 *Let B be a space. A* ***space over*** *B is a map $f : X \to B$. If $f : X \to B$ and $g : Y \to B$ are spaces over B, then a map $\alpha : X \to Y$ is said to be a* ***map over*** *B if it makes the following triangle commute*

$$\begin{tikzcd} X \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B. & \end{tikzcd} \tag{2.1}$$

There is a well-defined composition of maps over B which is induced from the composition in Top. Moreover identity functions are maps over B. Hence the spaces and maps over B form a category which we denote Top/B and call the ***category of spaces over*** *B.* □

We call B the **base** space. We call a space over B an **overspace** when repeated reference to B becomes clunky and write (X, f) to denote a given overspace $f : X \to B$. The map f itself is said to be the **structure map** of (X, f). We denote by $X_b = f^{-1}(b)$ the **fibre** over a point $b \in B$. The morphisms in Top/B are sometimes written $\alpha : X \to_B Y$, and are called **fibrewise** maps, or sometimes **fibre preserving** maps.

Example 2.1

1) If $B = \emptyset$, then $Top/\emptyset$ has a single object and morphism. For this reason we will assume in the sequel that B is nonempty unless specifically stated otherwise.

2) If $B = *$, then $Top/* \cong Top$.

3) Let $B = \mathcal{S} = \{u, c\}$ be the Sierpinski space, where $\{u\} \subseteq \mathcal{S}$ is open. Given a space X, the assignment $f \mapsto f^{-1}(u)$ establishes a one-to-one correspondence between the continuous maps $X \to \mathcal{S}$ and the open sets of X. Thus the objects of $Top/\mathcal{S}$ are pairs (X, U) of a space X and a distinguished open subset $U \subseteq X$. A fibre preserving map $\alpha : (X, U) \to (Y, V)$ is a continuous map satisfying $\alpha^{-1}(V) = U$. □

There is a forgetful functor

$$Top/B \xrightarrow{U} Top, \qquad (X, f) \mapsto X \tag{2.2}$$

which takes an overspace to its domain. In the opposite direction we get the functor

$$Top \xrightarrow{B\times(-)} Top/B, \qquad K \mapsto (B \times K, pr_B) \tag{2.3}$$

which takes an unbased space K to the overspace $pr_B : B \times K \to B$. A free map $f : K \to L$ is sent to the product $1_B \times f$.

Proposition 2.1 *There is an adjunction*

$$\begin{tikzcd} Top/B \arrow[r, bend left, "U"] \arrow[r, phantom, "\perp"] & Top. \arrow[l, bend left, "B\times(-)"] \end{tikzcd} \tag{2.4}$$

In particular, there are bijections

$$Top\,(U(X, f), K) \cong Top/B\big((X, f), (B \times K, pr_B)\big) \tag{2.5}$$

which are natural in both variables.

Proof Let $f : X \to B$ be a space over B and K a space. Then a map $\alpha : X \to B \times Y$ (in Top) is determined by a pair of maps $X \to B$ and $X \to K$. If α is a map over B, then the first of these maps must be equal to f. Thus α is completely determined (in Top/B) by its second component, which can be any map (in Top). In this way we get the bijection (2.5) whose naturality is easy to see. ∎

Next we give some elementary properties of the category Top/B. Proofs are left for the reader, although for completeness we do give full statements.

Proposition 2.2 *Let B be a space and consider the slice category Top/B.*

1) *The identity $id_B : B \xrightarrow{=} B$ is a terminal object in B.*

2) *The category Top/B has products. The product of $X \to B$ and $Y \to B$ is the topological pullback $X \times_B Y$ equipped with the canonical map to B.*

3) *The category Top/B has pullbacks. Given a diagram of spaces over B*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "f"'] & Z \arrow[d] & Y \arrow[l, "\beta"'] \arrow[dl, "g"] \\ & B & \end{tikzcd} \tag{2.6}$$

we form the topological pullback $X \times_Z Y$. The projection maps $X \times_Z Y \to X$ and $X \times_Z Y \to Y$ are maps over B, and if $u : U \to X$ and $v : U \to Y$ are maps over B satisfying $fu = gv$, then we may check directly that the unique map they specify $U \to X \times_Z Y$ is a map over B. In particular $X \times_Z Y$ is the pullback in Top/B of $(X, f), (Y, g)$.

4) *More generally, Top/B has all limits. See Borceux, [1] Pr. 2.8.2, for instance. Tracing through Borceux's proof we see that the limit of a diagram $\mathcal{D} \xrightarrow{F} Top/B$ may be calculated as follows. Let $\mathcal{D}_\infty$ be the category $\mathcal{D}$ with an extra terminal object ∞ adjoined (so that in particular every object in $\mathcal{D}_\infty$ admits a unique morphism to ∞). Then F defines a functor $\mathcal{D}_\infty \xrightarrow{F_\infty} Top$ by setting $F_\infty|_\mathcal{D} = F$ and $F_\infty(\infty) = B$. The limit of F_∞ in Top comes furnished with a canonical map to B, and so becomes a space over B. It also has canonical maps to each $F(d)$, $d \in \mathcal{D}$, and checking directly we see that this limit is also a limit for F in Top/B.* ∎

Proposition 2.3 *Let B be a space.*

1) *The unique map $\emptyset \to B$ is an initial object in Top/B.*

2) *The category Top/B has coproducts. The coproduct of $f : X \to B$ and $g : Y \to B$ is $(f, g) : X \sqcup Y \to B$.*

3) The category Top/B has pushouts. The pushout in Top/B of the top maps in a diagram

$$\begin{tikzcd} X \arrow[dr, "f"'] & Z \arrow[l, "\alpha"'] \arrow[r, "\beta"] \arrow[d] & Y \arrow[dl, "g"] \\ & B & \end{tikzcd} \tag{2.7}$$

is $(f, g) : X \cup_Z Y \to B$.

4) The category Top/B has all colimits. The colimit of a diagram $\mathcal{D} \xrightarrow{F} Top/B$ is the colimit in Top of the composition $\mathcal{D} \xrightarrow{F} Top/B \xrightarrow{U} Top$. It admits a canonical map to B. According to Proposition 2.1 U is a left adjoint, and thus the statement here follows from (the dual of) Borceux's Proposition 3.2.2 [1]. ∎

We will use the discussion of fibrewise pullbacks above to produce another interesting adjunction. Fix the space B and let $\theta : B \to C$ be a map. Then there is a **push forwards functor**

$$Top/B \xrightarrow{\theta_*} Top/C, \qquad (X, f) \mapsto (X, \theta f) \tag{2.8}$$

which sends $f : X \to B$ to the composite $\theta f : X \to C$. In the other direction there is a **pullback functor**

$$Top/C \xrightarrow{\theta^*} Top/B, \qquad (Y, g) \mapsto (\theta^* Y, p_g) \tag{2.9}$$

which sends $g : Y \to C$ to the canonical projection p_g from the pullback $\theta^* Y = B \times_C Y$ onto B.

Proposition 2.4 *Let $\theta : B \to C$ be a map. Then there is an adjunction*

$$\begin{tikzcd} Top/B \arrow[r, bend left, "\theta_*"] \arrow[r, phantom, "\perp"] & Top/C. \arrow[l, bend left, "\theta^*"] \end{tikzcd} \tag{2.10}$$

In particular the push forward functor θ_* is left adjoint to the pull back functor θ^*.

Proof Let $f : X \to B$ be a space over B and $\alpha : \theta_* X \to Y$ a map over C to an overspace $g : Y \to C$. Then $g\alpha = \theta f$, so there is a unique map $(f, \alpha) : X \to \theta^* Y$ into the pullback. This gives us a function $Top/C(\theta_* X, Y) \to Top/B(X, \theta^* Y)$ which we argue is a bijection using the uniqueness of the induced maps (f, α). Moreover, using the naturality of the pullback construction we see that this function is natural in both variables. ∎

We need to make one last construction before discussion homotopy in Top/B. Let K be a space and define a functor

$$Top/B \xrightarrow{(-)\otimes K} Top/B, \qquad (X, f) \mapsto (X, f) \otimes K = (X \times K, f_K) \tag{2.11}$$

which takes an overspace $f : X \to B$ to the composite $f_K : X \times K \xrightarrow{pr_1} X \xrightarrow{f} B$. Notice that there is a bijection $Top/B\big((X, f), (Y, g) \otimes K\big) \cong Top/B\big((X, f), (Y, g)\big) \times Top(U(X, f), K)$. This construction also functorial in the second variable, and a continuous map $K \to K'$ induces a natural transformation $\otimes K \Rightarrow \otimes K'$.

2.1 Homotopy in Top/B

To define homotopy in Top/B we need a cylinder object. This is supplied by the functor $(-) \otimes I$. In particular, given an overspace (X, f), for $t \in I$, each inclusion $in_t : X \hookrightarrow X \otimes I$, $x \mapsto (x, t)$, is a map over B. Similarly the projection $pr_X : X \otimes I \to X$, $(x, t) \mapsto x$, is a map over B.

Definition 2 *Let B be a space. A* ***fibrewise homotopy****, or* ***homotopy over*** *B, is a map $H : (X, f) \otimes I \to_B (Y, g)$ over B. The composites $H_t = H \circ in_t : X \to_B Y$, $t \in I$, are maps over B. If $\alpha = H_0$ and $\beta = H_1$, then we write $H : \alpha \simeq_B \beta$.* $\square$

Thus H is a map $X \times I \to Y$ satisfying $gH_t(x) = f(x)$ for all $x \in X, t \in I$. In particular H_t is a map over B at each time $t \in I$. Fibrewise homotopy is an equivalence relation on the set of fibrewise maps $(X, f) \to_B (Y, g)$ and we write

$$[(X, f), (Y, g)]_B \tag{2.12}$$

for the set of fibrewise homotopy classes. The relation respects compositions, and in particular given fibrewise maps $\beta : (Y, g) \to (Z, h)$ and $\alpha : (W, e) \to (X, f)$, there are induced maps

$$\alpha^* : [(X, f), (Y, g)]_B \to [(W, e), (Y, g)]_B, \qquad \beta_* : [(X, f), (Y, g)]_B \to [(X, f), (Z, h)]_B. \tag{2.13}$$

We form the **homotopy category** $hTop/B = h(Top/B)$ in the standard way. Note that this category should not be confused with the category $(hTop)/B$.

Definition 3 *A map over B is said to be a* ***fibrewise homotopy equivalence****, or* ***homotopy equivalence over*** *B, if it is invertible up to homotopy over B. A space $X \to B$ over B is said to be* ***shrinkable*** *if it is homotopy equivalent over B to the identity $id_B : B \xrightarrow{=} B$.* $\square$

Example 2.2 Let B be a fixed space.

1) If $X \to B$ is a space over B, then the projection $pr_X : X \otimes I \to_B X$ is a homotopy equivalence over B.

2) The evaluation $ev_0 : B^I \to B$, $l \mapsto l(0)$, is a shrinkable space over B.

3) If $B = I$ and $X = I \times \partial I \cup 0 \times I$ with $f : X \to B$ the projection onto the first factor, then f is a homotopy equivalence in Top, but f is not shrinkable. $\square$

Proposition 2.5 *Let $\theta : B \to C$ be a map in Top. Then each of the functors in the the pushforwards-pullback adjunction $\theta_* : Top/B \rightleftarrows Top/C : \theta^*$ preserves fibrewise homotopies. That is, if $\alpha \simeq_B \beta$, then $\theta_*\alpha \simeq_C \theta_*\beta$, whilst if $\gamma \simeq_C \delta$, then $\theta^*\gamma \simeq_B \theta^*\delta$.*

Proof Both functors preserve fiberwise cylinders. That is, if $X \to B$ is a space over B and $Y \to C$ a space over C, then

$$\theta_*(X \otimes I) = (\theta_* X) \otimes I, \qquad \theta^*(Y \otimes I) = (\theta^* Y) \otimes I. \tag{2.14}$$

Moreover both these identifications preserve the inclusions $in_a : X \hookrightarrow_B X \otimes I$ and $in_a : Y \hookrightarrow_C Y \otimes I$. This makes it clear that a homotopy $\alpha \simeq_B \beta$ over B is pushed forwards to a homotopy $\theta_*\alpha \simeq_C f_*\beta$ over C, whilst a homotopy $\gamma \simeq_C \delta$ over C is pulled back to a homotopy $\theta^*\gamma \simeq_B \theta^*\delta$ over B. ∎

Corollary 2.6 *The pullback functor θ^* induced by a fixed map $\theta : B \to C$ is homotopical. In particular, if $X \to C$ is homotopy equivalent to $Y \to C$ over C and $\theta : B \to C$ is a map in Top, then $\theta^*X \to B$ is homotopy equivalent to $\theta^*Y \to B$ over B.* ∎

The study of pullback functors with relation to fibrations is taken up in the next section.

Another question which will be addressed is the difference between fibrewise and ordinary homotopy theory. If a map $X \xrightarrow{\simeq}_B Y$ over B is a homotopy equivalence in Top/B, then it is also homotopy equivalence in Top after forgetting structure. On the other hand, if a fibrewise map $X \to_B Y$ is given, which happens to be a homotopy equivalence in Top, then is it possible that it is also a homotopy equivalence in Top/B? In general we must answer this in the negative.

Example 2.3 The triangle

$$\begin{tikzcd} \{0\} \arrow[rr, "\subseteq"] \arrow[dr, "1"'] & & \mathbb{R} \arrow[dl, "exp"] \\ & S^1 & \end{tikzcd} \tag{2.15}$$

commutes strictly and the inclusion $\{0\} \hookrightarrow \mathbb{R}$ is a homotopy equivalence. However there is no map from $\mathbb{R}$ to $\{0\}$ over S^1 (indeed, exp is surjective). Hence the diagram does not display a fibrewise homotopy equivalence over S^1. □

In section 4 we give the statement 4.5, that if $\alpha : X \to_B Y$ is a map over B which is a homotopy equivalence in Top, and *both* projections $X, Y \to B$ are fibrations, then α is actually homotopy equivalence over B.

3 The Homotopy Theorem

Let $p : E \to B$ be a fibration and X a space. Fix maps $f, g : X \to B$ and form the pullbacks

$$\begin{tikzcd} f^*E \arrow[r, "\hat{f}"] \arrow[d, "p_f"'] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} g^*E \arrow[r, "\hat{g}"] \arrow[d, "p_g"'] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d, "p"] \\ X \arrow[r, "g"] & B. \end{tikzcd} \tag{3.1}$$

Assume that f, g are homotopic. Following [2] § 5.6, we will show in the following paragraphs how a choice of homotopy $G : f \simeq g$ may be used to get a well-defined homotopy class of map over X

$$\theta_G : f^*E \to g^*E. \tag{3.2}$$

The construction is as follows. First we apply the HLP to the diagram

$$\begin{tikzcd}
f^*E \arrow[rr, "\hat{f}"] \arrow[d, "in_0"'] & & E \arrow[d, "p"] \\
f^*E \times I \arrow[r, "p_f \times 1"'] \arrow[urr, dashed, "\widetilde{G}"] & X \times I \arrow[r, "G"] & B
\end{tikzcd} \tag{3.3}$$

to get a map $\widetilde{G} : f^*E \times I \to E$ satisfying $p\widetilde{G}_1 = gp_f$. From this we get a map $\widetilde{\theta}_G : f^*E \to g^*E$ as that induced by the pullback

$$\begin{tikzcd}
f^*E \arrow[dr, dashed, "\widetilde{\theta}_G"] \arrow[drr, bend left, "\widetilde{G}_1"] \arrow[ddr, bend right, "p_f"'] & & \\
& g^*E \arrow[r, "\hat{g}"] \arrow[d, "p_g"] & E \arrow[d, "p"] \\
& X \arrow[r, "g"] & B.
\end{tikzcd} \tag{3.4}$$

Note that $\widetilde{\theta}_G$ depends not only on G but also on the choice of lift $\widetilde{G}$ and the notation here is designed to reflect this. It is a consequence of Lemma 3.1 below, however, that the homotopy class of this map over X is independent of the choice of lift. Thus we set

$$\theta_G = [\widetilde{\theta}_G] \in [f^*E, g^*E]_X. \tag{3.5}$$

The fact that this is well-defined in fact follows from a more general result. To set it up assume given a second homotopy $H : f \simeq g$. Choose a lift $\widetilde{H} : f^*E \times I \to E$ as in 3.3 and use it to construct a corresponding map $\widetilde{\theta}_H : f^*E \to g^*E$.

Lemma 3.1 *A track homotopy $\psi : G \sim H$ induces a homotopy $\Psi : \widetilde{\theta}_G \simeq_X \widetilde{\theta}_H$ over X.*

Proof We work with the diagram

$$\begin{tikzcd}
(f^*E \times 0 \times I) \cup (f^*E \times I \times 0) \cup (f^*E \times I \times 1) \arrow[rr, "\widehat{f} \cup \widetilde{G} \cup \widetilde{H}"] \arrow[d] & & E \arrow[d, "p"] \\
f^*E \times I \times I \arrow[r, "p_f \times 1 \times 1"'] \arrow[urr, dashed, "\widetilde{\psi}"] & X \times I \times I \arrow[r, "\psi"] & B.
\end{tikzcd} \tag{3.6}$$

Since

- $\psi(x, 0, s) = f(x)$
- $\psi(x, t, 0) = G(x, t)$
- $\psi(x, t, 1) = H(x, t)$

the solid part of the diagram commutes. As the vertical arrow on the left is both a homotopy equivalence and cofibration, the dotted arrow can be filled in to yield a map $\widetilde{\psi}$ satisfying

- $p\widetilde{\psi}(e,t,s) = \psi(p_f(e),t,s)$
- $\widetilde{\psi}(e,0,s) = \hat{f}(e)$
- $\widetilde{\psi}(e,t,0) = \widetilde{G}(e,t)$
- $\widetilde{\psi}(e,t,1) = \widetilde{H}(e,t)$.

Let $\Psi' : f^*E \times I \to E$ be the map $\Psi'(e,t) = \widetilde{\psi}(e,1,t)$. Then the following diagram commutes

$$\begin{tikzcd} f^*E \times I \arrow[dr, dashed, "\Psi"] \arrow[drr, bend left, "\Psi'"] \arrow[ddr, bend right, "p_f \circ pr_1"'] & & \\ & g^*E \arrow[r, "\hat{g}"] \arrow[d, "p_g"] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d, "p"] \\ & X \arrow[r, "g"] & B. \end{tikzcd} \tag{3.7}$$

and induces the homotopy Ψ. Since $\Psi'|_{f^*E\times 0} = \widetilde{G}_1$ we get from the uniqueness of the map into the pullback that $\Psi_0 = \widetilde{\theta}_G$. Similarly $\Psi'|_{f^*E\times 1} = \widetilde{H}_1$ implies that $\Psi_1 = \widetilde{\theta}_H$. ∎

If we take $G = H$, then $\widetilde{G}, \widetilde{H}$ are lifts of the same homotopy. Taking ψ be the identity track on G and applying the lemma thus establishes the independence of $\widetilde{\theta}_G$ on the choice of lift up to homotopy.

Proposition 3.2 *The class $\theta_G = [\widetilde{\theta}_G] \in [f^*E, g^*E]_X$ is well-defined.* ∎

Now fix a third map $h : X \to B$ and form a third pullback

$$\begin{tikzcd} h^*E \arrow[r, "\hat{h}"] \arrow[d, "p_h"'] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d, "p"] \\ X \arrow[r, "h"] & B. \end{tikzcd} \tag{3.8}$$

Assume that $H : g \simeq h$ is a homotopy and let $\theta_H \in [g^*E, h^*E]_X$ be the corresponding class.

Lemma 3.3 *It holds that*

$$\theta_{G+H} = \theta_H \theta_G \in [f^*E, h^*E]_X. \tag{3.9}$$

Proof Fix lifts $\widetilde{G}$ and $\widetilde{H}$ as in the diagrams

$$\begin{tikzcd} f^*E \arrow[rr, "\hat{f}"] \arrow[d, "in_0"'] & & E \arrow[d, "p"] \\ f^*E \times I \arrow[r, "p_f \times 1"'] \arrow[urr, dashed, "\widetilde{G}"] & X \times I \arrow[r, "G"] & B \end{tikzcd} \qquad \begin{tikzcd} g^*E \arrow[rr, "\hat{g}"] \arrow[d, "in_0"'] & & E \arrow[d, "p"] \\ g^*E \times I \arrow[r, "p_g \times 1"'] \arrow[urr, dashed, "\widetilde{H}"] & X \times I \arrow[r, "H"] & B. \end{tikzcd} \tag{3.10}$$

and use them to define maps $\widetilde{\theta}_G : f^*E \to g^*E$ and $\widetilde{\theta}_H : g^*E \to h^*E$ as in (3.4). Note that

$$\widetilde{H}_0 \widetilde{\theta}_G = \hat{g}\widetilde{\theta}_G = \widetilde{G}_1 \tag{3.11}$$

so the homotopy $\widetilde{G}+\widetilde{H}\widetilde{\theta}_G : E_f \times I \to E$ is defined. Then $(\widetilde{G}+\widetilde{H}\widetilde{\theta}_G)_0 = \widetilde{G}_0 = \hat{f}$ and

$$p(\widetilde{G}+\widetilde{H}\widetilde{\theta}_G) = p\widetilde{G}+p\widetilde{H}\widetilde{\theta}_G = Gp_f + Hp_g\widetilde{\theta}_G = Gp_f + Hp_f = (G+H)p_f. \tag{3.12}$$

It follows that the homotopy $\widetilde{G}+\widetilde{H}\widetilde{\theta}_G$ completes the dotted lift in the next diagram

$$\begin{tikzcd} f^*E \arrow[rr, "\hat{f}"] \arrow[d, "in_0"'] & & E \arrow[d, "p"] \\ f^*E \times I \arrow[r, "p_f\times 1"'] \arrow[urr, dashed, "\widetilde{G}+\widetilde{H}\widetilde{\theta}_G"] & X \times I \arrow[r, "G+H"] & B \end{tikzcd} \tag{3.13}$$

and so is suitable for defining the class $\theta_{G+H} \in [f^*E, h^*E]_X$.

On the other hand, we can be more explicit about this class. In particular it is defined by means of a map $\widetilde{\theta}_{G+H}$, which is itself constructed from the $\widetilde{G}+\widetilde{H}\widetilde{\theta}_G$ as in (3.4). Since

$$(\widetilde{G}+\widetilde{H}\widetilde{\theta}_G)_1 = \widetilde{H}_1\widetilde{\theta}_G \tag{3.14}$$

we see that the

$$\widetilde{\theta}_{G+H} = \widetilde{\theta}_H\widetilde{\theta}_G \tag{3.15}$$

and so get the claim. ∎

The lemma shows how the construction interacts with vertical composition of homotopies. We show next that it also interacts pleasantly well also with the horizontal composition. For this we will need to take some care with notation. Assume that $f \simeq g : X \to B$ are as above, and that we now have homotopic maps $k \simeq l : Y \to X$. Forming the iterated pullback diagram

$$\begin{tikzcd} k^*(f^*E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & f^*E \arrow[r, "\hat{f}"] \arrow[d, "p_f"'] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d, "p"] \\ Y \arrow[r, "k"] & X \arrow[r, "f"] & B \end{tikzcd} \tag{3.16}$$

there is a canonical identification $k^*f^*E \cong (fk)^*E$. Given homotopies $G : f \simeq g$ and $H : k \simeq l$, their composition gives a map

$$\theta_{GH} : k^*f^*E \cong (fk)^*E \to (gl)^*E \cong l^*g^*E \tag{3.17}$$

which is well-defined up to homotopy. On the other hand, there is also the map $\theta_G : f^*E \to g^*E$, which by pullback induces maps

$$k^*\theta_G : k^*f^*E \to k^*g^*E, \qquad l^*\theta_G : l^*f^*E \to l^*g^*E \tag{3.18}$$

and

$$f_*\theta_H : k^*f^*E \to l^*f^*E, \qquad g_*\theta_H : k^*f^*E \to l^*f^*E \tag{3.19}$$

where the f_*, g_* is notation only to indicate the domains of the maps (3.2).

These maps compose to give three homotopy classes $k^*f^*E \to l^*g^*E$ over X. We apply the interchange law

$$Gk + gH \sim GH \sim fH + Gl \tag{3.20}$$

and appeal to Lemmas 3.1, 3.3 to get the following

Proposition 3.4 *With the notation above, the diagram*

$$\begin{tikzcd} k^*f^*E \arrow[r, "k^*\theta_G"] \arrow[d, "f_*\theta_H"'] & k^*g^*E \arrow[d, "g_*\theta_H"] \\ l^*f^*E \arrow[r, "l^*\theta_H"] & l^*g^*E \end{tikzcd} \tag{3.21}$$

commutes up to homotopy over X. Moreover both composition $k^*f^*E \to l^*g^*E$ are homotopic to the map θ_{GH}. ∎

3.1 Implications

The machinery developed so far this section is quite powerful. Here we put it to good use and collect some important statements which we will use in future. The main results of this section are the *homotopy theorem* for fibrations 3.5 and its corollary 3.6, the homotopy invariance of fibres.

Theorem 3.5 *If $p : E \to B$ is a fibration, and $f \simeq g : X \to B$ are homotopic maps, then there is a homotopy equivalence $f^*E \simeq_X g^*E$ over X.*

Proof A choice of homotopy $G : f \simeq g$ induces homotopy classes $\theta_G : f^*E \to g^*E$ and $\theta_{-G} : g^*E \to f^*E$. Using Lemma 3.1 we see that the trivial homotopy $1_f : f \simeq f$ induces the class of the identity on f^*E. We combine Lemmas 3.1 and 3.3 together with the equation $G - G \sim 1_f$ to get

$$\theta_{-G}\theta_G = \theta_{G-G} = \theta_{1_f} = [id_{f^*E}]. \tag{3.22}$$

Similarly we check that

$$\theta_G\theta_{-G} = \theta_{-G+G} = \theta_{1_g} = [id_{g^*E}]. \quad ∎ \tag{3.23}$$

If $b, c \in B$, then there are pullbacks

$$\begin{tikzcd} E_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d, "p"] \\ * \arrow[r, "b"] & B \end{tikzcd} \qquad \begin{tikzcd} E_c \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d, "p"] \\ * \arrow[r, "c"] & B. \end{tikzcd} \tag{3.24}$$

where E_b, E_c are the fibres of p over b, c, respectively. Assume that b, c lie in the same path component. Then a choice of path $\alpha : I \to B$ gives rise to a homotopy class

$$\theta_\alpha : E_b \to E_c \tag{3.25}$$

from the fibre of p over b to the fibre of p over c. According to Corollary 3.5 this map is a homotopy equivalence.

Corollary 3.6 *Let $p : E \to B$ be a fibration. All the fibres of p over points in the same path component of B have the same homotopy type. In particular, if B is path-connected, then all fibres of p are homotopy equivalent.* ∎

If $\beta : I \to B$ is a path from b to a third point c, then

$$\theta_{\alpha+\beta} = \theta_\beta \theta_\alpha \tag{3.26}$$

holds up to homotopy. In particular, if B is equipped with a basepoint $*$ and $F = p^{-1}(B)$ is the typical fibre of p, then there is a well-defined homomorphism

$$\pi_1 B \to \mathcal{E}(F), \qquad \alpha \mapsto \theta_{\alpha^{-1}} \tag{3.27}$$

where $\mathcal{E}(F)$ is the group, under composition, of homotopy classes of self-homotopy equivalences $F \to F$. We think of $\pi_1 B$ as acting 'up to homotopy' on F through this homomorphism. In particular $\pi_1 B$ acts on all homotopy, homology and cohomology groups of F.

Example 3.1 Here we will consider the universal covering $\gamma_n : S^n \to \mathbb{R}P^n$. This map is a fibration[1] and we will compute the action of $\pi_1 \mathbb{R}P^n$ on the discrete fibre $F = \mathbb{Z}_2$. Since the fibre is discrete, the action (3.27) will actually factor through one by homeomorphisms, so the conclusion should come as no surprise to anyone with some experience of covering spaces.

If $n = 1$ then there is a homeomorphism

$$S^1 \xrightarrow{\cong} \mathbb{R}P^1, \qquad t \mapsto [\cos(\pi t), \sin(\pi t)] \tag{3.28}$$

where $t \in S^1 \cong I/\partial I$. We'll focus on the $n \geq 2$ case below, but we need this map to do so. If $n \geq 2$ then $\mathbb{R}P^n$ is the adjunction space $\mathbb{R}P^n \cong \mathbb{R}P^{n-1} \cup_{\gamma_{n-1}} e^n$. An inductive application of the Seifert-Van Kampen Theorem shows that $\pi_1 \mathbb{R}P^n \cong \mathbb{Z}_2$ for $n \geq 2$ and is generated by the composite $S^1 \cong \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n$. Given (3.28) we represent this generator as a path by the map

$$\alpha : I \to \mathbb{R}P^n, \qquad t \mapsto [\cos(\pi t), \sin(\pi t), 0, \ldots, 0]. \tag{3.29}$$

Now recall the steps needed to define the map $\theta_\alpha : \mathbb{Z}_2 \to \mathbb{Z}_2$. We find the lift $\widetilde{\alpha}$ in the diagram (this is (3.3))

$$\begin{tikzcd} F = \mathbb{Z}_2 \arrow[rr] \arrow[d, "in_0"'] & & S^n \arrow[d, "p"] \\ \mathbb{Z}_2 \times I \arrow[r, "pr_2"'] \arrow[urr, dashed, "\widetilde{\alpha}"] & I \arrow[r, "\alpha"] & \mathbb{R}P^n \end{tikzcd} \tag{3.30}$$

and then set

$$\theta_\alpha = \widetilde{\alpha}_1. \tag{3.31}$$

This has values in the fibre, so we consider θ_α as a map $\mathbb{Z}_2 \to \mathbb{Z}_2$.

Now our choice of representative (3.29) makes it clear that we can choose $\widetilde{\alpha}$ to be the map

$$\widetilde{\alpha}(\pm 1, t) = (\pm \cos(\pi t), \pm \sin(\pi t), 0, \ldots, 0). \tag{3.32}$$

In particular, the action of $\theta_\alpha = \widetilde{\alpha}_1$ on $\mathbb{Z}_2$ is just multiplication by the generator.

As we mentioned above, it is not difficult to generalise this example to compute the action of $\pi_1 X$ on any covering space of X. □

[1] That covering spaces are fibrations will be discussed next lecture.

We end this section with another corollary of Theorem 3.5 which is unrelated to the proceeding discussion.

Corollary 3.7 *Let a pullback square*

$$\begin{tikzcd} f^*E \arrow[r, "\hat{f}"] \arrow[d, "p_f"'] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd} \tag{3.33}$$

be given. Assume that $p : E \to B$ is a fibration and $f : X \xrightarrow{\simeq} B$ is a homotopy equivalence. Then $\hat{f} : f^*E \to E$ is a homotopy equivalence.

Proof Choose a homotopy inverse $g : B \xrightarrow{\simeq} X$ to f. Then there is a homeomorphism between the following two pullback spaces

$$g^*(f^*E) \cong (gf)^*E \tag{3.34}$$

and according to Theorem 3.5, also a homotopy equivalence

$$(gf)^*E \simeq (id_B)^*E = E. \quad \blacksquare \tag{3.35}$$

4 Transport

Consider a diagram of spaces over B

$$\begin{tikzcd} X \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd} \tag{4.1}$$

where f is a map and p is a fibration. Assume that $F : f \simeq g : X \to B$ is a homotopy. We show in this paragraph how to use F to turn (4.1) into a diagram

$$\begin{tikzcd} X \arrow[rr, "F^{\#}\alpha"] \arrow[dr, "g"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd} \tag{4.2}$$

where $F^{\#}\alpha$ is a map defined up to homotopy over B. We show how the homotopy class of $F^{\#}\alpha$ over B depends only on α through its homotopy class over B. In this way our construction defines a function

$$F^{\#} : [(X, f), (E, p)]_B \to [(X, g), (E, p)]_B \tag{4.3}$$

called **transport along** F.

The construction is as follows. Starting with (4.1) we apply the HLP to the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "in_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"] \arrow[ur, dashed, "\widetilde{F}"] & B \end{tikzcd} \tag{4.4}$$

to find a homotopy $\widetilde{F} : X \times I \to E$ with $\widetilde{F}_0 = \alpha$ and $p\widetilde{F} = F$. Set

$$F^{\#}\alpha = \widetilde{F}_1 : X \to E. \tag{4.5}$$

We show next that the homotopy class of this map over B is independent of the choice of lift $\widetilde{F}$, as well as that it depends only on F through its track homotopy class. The constructions are formally very similar to the last section, so details are sketched.

If $\psi : F \sim G$ is a track homotopy, then we find a lift $\widetilde{G} : X \times I \to E$ and apply the covering homotopy lifting property to the following diagram

$$\begin{tikzcd} X \times I \times 0 \cup X \times 0 \times I \times I \cup X \times I \times 1 \arrow[r, "\widetilde{F}_1 \cup \alpha \cup \widetilde{G}_1"] \arrow[d] & E \arrow[d, "p"] \\ X \times I \times I \arrow[r, "\psi"] \arrow[ur, dashed, "\widetilde{\psi}"] & B \end{tikzcd} \tag{4.6}$$

to get $\widetilde{\psi} : X \times I \times I \to E$. Then $(x,t) \mapsto \widetilde{\psi}(x,1,t)$ is a homotopy over B

$$F^{\#}\alpha = \widetilde{F}_1 \simeq_B \widetilde{G}_1 = G^{\#}\alpha. \tag{4.7}$$

Next we assume that $F : f \simeq g$ and $G : g \simeq h$ are homotopies of maps $X \to B$. We construct transport maps

$$F^{\#} : [(X,f),(E,p)]_B \to [(X,g),(E,p)]_B, \quad G^{\#} : [(X,f),(E,p)]_B \to [(X,h),(E,p)]_B \tag{4.8}$$

and would like to compare the composite $G^{\#} \circ F^{\#}$ with

$$(F+G)^{\#} : [(X,f),(E,p)]_B \to [(X,h),(E,p)]_B. \tag{4.9}$$

Choose lifts $\widetilde{F}, \widetilde{G}$ as in the following diagrams

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "in_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"] \arrow[ur, dashed, "\widetilde{F}"] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\widetilde{F}_1"] \arrow[d, "in_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "G"] \arrow[ur, dashed, "\widetilde{G}"] & B. \end{tikzcd} \tag{4.10}$$

Then $G^{\#}F^{\#}(\alpha) = G^{\#}([\widetilde{F}_1]) = [\widetilde{G}_1]$. On the other hand $\widetilde{F} + \widetilde{G}$ is defined and provides a lift in the following diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "in_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F+G"] \arrow[ur, dashed, "\widetilde{F}+\widetilde{G}"] & B. \end{tikzcd} \tag{4.11}$$

Thus
$$(F+G)^{\#}(\alpha) = [(\widetilde{F}+\widetilde{G})_1] = [\widetilde{G}_1]. \tag{4.12}$$
In particular
$$(F+G)^{\#} = G^{\#} \circ F^{\#}. \tag{4.13}$$

Summarising the results of this section we present the following.

Proposition 4.1 *Let $F : f \simeq g$ be a homotopy of maps $X \to B$. Assume that $p : E \to B$ is a fibration. Then the transport along F is a well-defined bijection*

$$F^{\#} : [(X,f),(E,p)]_B \to [(X,g),(E,p)]_B \tag{4.14}$$

which depends only on the track homotopy class of F. If $G : g \simeq h$ is a second homotopy of maps $X \to B$, then the transport functors $F^{\#}$, $G^{\#}$ compose as

$$G^{\#} \circ F^{\#} = (F+G)^{\#}. \tag{4.15}$$

Proof The argument culminating in (4.7) shows that $F^{\#}$ is well-defined and depends only on the track homotopy class of F. The equation (4.15) is explained in the paragraph proceeding (4.13). To see that $F^{\#}$ is bijective we use

$$F - F \sim 1_f \tag{4.16}$$

to get
$$F^{\#} \circ (-F)^{\#} = (F-F)^{\#} = (1_f)^{\#} = id \tag{4.17}$$
and similarly $-F + F \sim 1_G$ to get $(-F)^{\#} \circ F^{\#} = id$. ∎

4.1 Implications

The main result of this section is Theorem 4.4. We start more generally by considering the following situation

$$\begin{tikzcd} X \arrow[rrr, "\alpha f"] \arrow[dr, "f"'] & & & E \arrow[dll, "p"] \\ & Y \arrow[urr, "\alpha"] \arrow[dr, "k"'] & & \\ & & B & \end{tikzcd} \tag{4.18}$$

where we assume given homotopy $F : f \simeq g$ of maps $X \to Y$.

Lemma 4.2 *The following diagram commutes*

$$\begin{tikzcd} & {[(Y,k),(E,p)]_B} \arrow[dl, "f^*"'] \arrow[dr, "g^*"] & \\ {[(X,kf),(E,p)]_B} \arrow[rr, "(kF)^{\#}"] & & {[(X,kg),(E,p)]_B}. \end{tikzcd} \tag{4.19}$$

Proof Let $\alpha \in [(Y,k),(E,p)]_B$. From the definition we have $(kF)^{\#} f^*(\alpha) = (kF)^{\#}(\alpha f) = [\widetilde{kF}_1]$, where $\widetilde{kF}$ is some lifting homotopy as in the following diagram

$$\begin{tikzcd} X \arrow[r, "\alpha f"] \arrow[d, "in_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "kF"] \arrow[ur, dashed, "\widetilde{kF}"] & B. \end{tikzcd} \tag{4.20}$$

If we set $\widetilde{kF} = \alpha F$, then

$$p\widetilde{kF} = p(\alpha F) = kf, \qquad \widetilde{kF}_0 = \alpha f. \tag{4.21}$$

So choosing $\widetilde{kF}$ in this way we have

$$(kF)^{\#} f^*(\alpha) = [\widetilde{kF}_1] = [\alpha g] = g^*[\alpha] \tag{4.22}$$

which is what we needed to show. ∎

Proposition 4.3 *In the situation of* (4.18)*, assume that p is a fibration and f is an ordinary homotopy equivlance. Then*

$$f^* : [(Y,k),(E,p)]_B \to [(X,kf),(E,p)]_B \tag{4.23}$$

is bijective.

Proof Let $g : Y \to X$ be a homotopy inverse to f and consider the sequence of sets

$$[(Y,k),(E,p)]_B \xrightarrow{f^*} [(Y,kf),(E,p)]_B \xrightarrow{g^*} [(Y,kfg),(E,p)]_B \xrightarrow{f^*} [(X,kfgf),(E,p)]_B. \tag{4.24}$$

Choose a homotopy $F : fg \simeq id_Y$ and apply Lemma 4.2 to get a commutative diagram

$$\begin{tikzcd} & {[(Y,k),(E,p)]_B} \arrow[dl, "(fg)^*"'] \arrow[dr, "(id_Y)^*=1"] & \\ {[(Y,kfg),(E,p)]_B} \arrow[rr, "(kF)^{\#}"] & & {[(Y,k),(E,p)]_B}. \end{tikzcd} \tag{4.25}$$

Then $g^* f^* = (fg)^*$ is bijective, which implies that f^* is injective. Choosing a homotopy $gf \simeq id_X$ we form an identical argument to show that $f^* g^* = (gf)^*$ is bijective. This second equation implies that f^* is surjective, and so gives us the proposition. ∎

Theorem 4.4 *Let $p : E \to B$, $q : F \to B$ be fibrations and $\alpha : E \to F$ a map over B*

$$\begin{tikzcd} E \arrow[rr, "\alpha"] \arrow[dr, "p"'] & & F \arrow[dl, "q"] \\ & B. & \end{tikzcd} \tag{4.26}$$

Assume that α is an ordinary homotopy equivalence. Then α is a homotopy equivalence over B.

Proof According to proposition 4.3, the induced map

$$\alpha^* : [(F,q),(E,p)]_B \to [(E,p),(E,p)]_B \tag{4.27}$$

is bijective. In particular there is map $\beta : F \to E$ over B such that

$$\alpha^*[\beta] = [\beta\alpha] = [id_E] \tag{4.28}$$

which implies that β is both

i) a left homotopy inverse to α over B

ii) a two-sided ordinary homotopy inverse to α.

It follows from the last point that

$$\beta^* : [(F,q),(F,q)]_B \to [(E,p),(F,q)] \tag{4.29}$$

is bijective so there thus exists a map $\gamma : E \to F$ over B such that

$$\beta^*[\gamma] = [\gamma\beta] = [id_F]. \tag{4.30}$$

But this shows that β is has both left- and right-homotopy inverse over B, which implies that β is a homotopy equivalence over B. Thus by the uniqueness of homotopy inverses in Top_B, we must have $\alpha \simeq_B \gamma$, so that α is a homotopy inverse to β in Top_B. ∎

A surjective map $f : X \to B$ is said to be **weakly shrinkable** if it is a homotopy equivalence which admits a strict section $s : B \to X$. The map f is said to be **shrinkable** if it is weakly shrinkable and there is a homotopy $F : sp \simeq id_E$ over B. This means that for each $b \in B$, the homotopy F satisfies $pF_t(e) = p(e)$ for all $t \in I$. The reader can check that equations these are just the details of Definition 3 spelled out. Shrinkable is the notion dual to that of deformation retraction.

Corollary 4.5 *Let $p : E \to B$ a fibration and a homotopy equivalence. Then p is shrinkable.*

Proof Choose a homotopy inverse $s' : B \to E$ to p and a homotopy $F : ps' \simeq id_B$. The last equation shows that the image of p meets each path component of B, and this implies that p is surjective, since it is a fibration. Now apply the HLP to the diagram

$$\begin{tikzcd} B \arrow[r, "s'"] \arrow[d, "in_0"'] & E \arrow[d, "p"] \\ B \times I \arrow[r, "F"] \arrow[ur, dashed, "\widetilde{F}"] & B \end{tikzcd} \tag{4.31}$$

and set $s = \widetilde{F}_1 : B \to E$. Then $ps = id_B$, which shows that p is weakly shrinkable. Next observe the following diagram of spaces over B

$$\begin{tikzcd} B \arrow[rr, "s"] \arrow[dr, equal] & & E \arrow[dl, "p"] \\ & B. & \end{tikzcd} \tag{4.32}$$

The map s satisfies $sp \simeq s'p \simeq id_E$, so is an ordinary homotopy equivalence. But in light of Theorem 4.4 this shows that s is actually a homotopy equivalence over B. ∎

The following is another corollary we can draw from 4.4. The reader should compare it with the statement 3.7.

Corollary 4.6 *Let a pullback square*

$$\begin{tikzcd} f^*E \arrow[r, "\hat{f}"] \arrow[d, "p_f"'] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd} \tag{4.33}$$

be given. Assume that $p : E \to B$ is both a fibration and a homotopy equivalence. Then so is $p_f : f^*E \to X$.

Proof It was shown in *Fibrations I* that p_f is a fibration so we need only show that it is a homotopy equivalence. Following Theorem 4.4 we find a section s of p, which we view as a map over B of the form $s : (B, id_B) \to (E, P)$. Then the map $f : X \to B$ induces a pullback functor $f^* : Top/B \to Top/X$ and $f^*(s)$ is a map

$$f^*(s) : f^*(B, id_B) \cong (X, id_X) \to (f^*E, p_f) \tag{4.34}$$

over X. That is $f^*(s)$ is a section of p_f. It was shown in Corollary 2.6 that the pullback functor f^* is homotopical, so the fibrewise homotopy $sp \simeq_B id_E$ is pulled back to homotopy $f^*(s)p_f \simeq_X id_{f^*E}$ over X. ∎

5 Proof of the Fundamental Lifting Property Completed.

Proof of 1.1: p is a homotopy equivalence We need to show that if p is a fibration and a homotopy equivalence, then the dotted arrow may be completed in any diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] \arrow[ur, dashed, "h"] & B \end{tikzcd} \tag{5.1}$$

in which j is a closed cofibration. To proceed choose a strict section $s : B \to E$ and a homotopy $F : sp \simeq id_E$ over B, both of which exist according to Corollary 4.5. Now apply the covering homotopy lifting property to the diagram

$$\begin{tikzcd} X \times 0 \cup A \times I \arrow[rr, "sg \cup F(g \times 1)"] \arrow[d] & & E \arrow[d, "p"] \\ X \times I \arrow[r, "pr_1"'] \arrow[urr, dashed, "\widetilde{H}"] & X \arrow[r, "g"] & B \end{tikzcd} \tag{5.2}$$

and let $h = \widetilde{H}_1$. Then $ph = g$ and $hj = F_1 g = g$, so we're done. ∎

6 The Mutal Characterisation of Cofibrations and Fibrations

The purpose of this section is to formulate a sort of converse to the Fundamental Lifting Property. This will be a formal characterisation of cofibrations and fibrations in terms of the existence of diagonal fillers in certain commutative spaces. Consequently it is possible to explore much of the theory of cofibrations and fibrations on a purely formal level. Our presentation follows Strøm's original paper [3]. Some implications of the Orthogonality Theorem 6.2 which are actually easier to prove in the formal setting are explored at the end of this section.

Definition 4 *Let $f : A \to X$ and $g : B \to Y$ be continuous maps. We say that f has the* ***left lifting property*** *(LLP) with respect to g, and that g has the* ***right lifting property*** *(RLP) with respect to f, if the dotted filler can be completed in any given commutative square of the form*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd} \tag{6.1}$$

Let $\mathcal{E}$ be a class of maps in Top. We say that f has the ***left lifting property*** *with respect to $\mathcal{E}$ if it has the left lifting property with respect to each map in $\mathcal{E}$. We say that g has the* ***right lifting property*** *with respect to $\mathcal{E}$ if it has the right lifting property with respect to each map in $\mathcal{E}$.* $\square$

Example 6.1

1) A map $g : B \to Y$ has the right lifting property with respect to $\emptyset \to *$ if and only if it is surjective. g has the right lifting property with respect to $S^0 \to *$ if and only if it is injective.

2) By definition a map $f : A \to X$ is a cofibration if and only if has the left lifting property with respect to the class of maps $\{e_0 : Y^I \to Y\}$.

3) By definition a map $g : B \to Y$ is a fibration if and only if it has the right lifting property with respect to the class of maps $\{in_0 : A \hookrightarrow A \times I\}$. $\square$

Lemma 6.1 *Let X be any space and $A \subseteq X$ be an inclusion. Let $E_A = A \times I \cup X \times (0, 1] \subseteq X \times I$ and let $\pi : E_A \to X$ be the projection onto the first factor. Then π is both a homotopy equivalence and a fibration.*

Proof It is easy to see that π is a homotopy equivalence and we leave this to the reader. To see that π is a fibration assume given a homotopy lifting problem

$$\begin{tikzcd} K \arrow[r, "v"] \arrow[d] & E_A \arrow[d, "\pi"] \\ K \times I \arrow[r, "H"] \arrow[ur, dashed] & X \end{tikzcd} \tag{6.2}$$

where K is some space. The map $\widetilde{H} : K \times I \to E$ given by

$$\widetilde{H}_t(k) = (H_t(k), t + (1-t)pr_2(v(k))) \tag{6.3}$$

is then a solution. $\square$

The following is the main result of the section. We call it the **Orthogonality Theorem**.

Theorem 6.2 *1) A map $f : A \to X$ is a closed cofibration if and only if has the left lifting property with respect to all maps which are both fibrations and homotopy equivalences.*

2) A map $f : A \to X$ is both a closed cofibration and a homotopy equivalence if and only if has the left lifting property with respect to all fibrations.

3) A map $g : B \to Y$ is a fibration if and only if has the right lifting property with respect to all maps which are both cofibrations and homotopy equivalences.

4) A map $g : B \to Y$ is both a fibration and a homotopy equivalence if and only if has the right lifting property with respect to all cofibrations.

Proof Each of the four forwards implications follows by applying the Fundamental Lifting Property 1.1, so we only prove the four reverse implications.

1) $\Leftarrow$ Assume that $f : A \to X$ has the LLP with respect to all fibrations which are homotopy equivalences. Then in particular f lifts against the class of maps $\{e_0 : Y^I \to Y\}$ so is a cofibration. This means that f is an embedding, and we will be done if we can show that it is closed.

For this let E_A and $\pi : E_A \to X$ be as in Lemma 6.1. Since f has the LLP with respect to π, the map $A \to E$, $a \mapsto (a, 0)$ has an extension $\widetilde{f} : X \to E$ which satisfies $\pi\widetilde{f} = id_X$. Then $A = \widetilde{f}^{-1}pr_2^{-1}(\{0\})$, so is closed.

2) $\Leftarrow$ It follows from part 1) that f must be a closed cofibration. To show that f is also a homotopy equivalence we work as follows. Using the assumptions we find a retraction $r : X \to A$ as the dotted filler in the diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[d] \\ X \arrow[r] \arrow[ur, dashed, "r"] & * . \end{tikzcd} \tag{6.4}$$

We then get a homotopy $H : id_X \simeq fr$ by using the LLP to find a filler for the diagram

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "f"'] & X^I \arrow[d, "e_{0,1}"] \\ X \arrow[r, "(id_X, fr)"'] \arrow[ur, dashed] & X \times X \end{tikzcd} \tag{6.5}$$

where c sends a point $a \in A$ to the constant path at $f(a) \in X$. It was shown in § 3 of *Fibrations I* that the start-end evaluation $e_{0,1}$ is a fibration.

3) $\Leftarrow$ Since for any space X, the inclusion $in_0 : X \hookrightarrow X \times I$ is a closed cofibration and a homotopy equivalence, we see that g is a fibration.

4) $\Leftarrow$ It follows similarly to 3) that g is a fibration. Applying the RLP to the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "s"] & Y \end{tikzcd} \tag{6.6}$$

we get a section s of g. Applying the RLP a second time, this time to

$$\begin{tikzcd} B \times 0 \cup B \times 1 \arrow[rr, "sg \cup id_B"] \arrow[d] & & B \arrow[d, "g"] \\ B \times I \arrow[r, "pr_1"] \arrow[urr, dashed, "H"] & B \arrow[r, "g"] & Y \end{tikzcd} \tag{6.7}$$

we get a homotopy $H : sg \simeq id_B$. In particular g is a homotopy equivalence. ■

Let $\mathcal{C}$ denote the class of all closed cofibrations, $\mathcal{F}$ the class of all fibrations, and $\mathcal{W}$ the class of all homotopy equivalences.

6.1 Implications

As mentioned above, many of the formal properties of cofibrations and fibrations can be obtained from purely formal manipulations based around the Orthogonality Theorem. For example we invite the reader to return to *Fibration I* and derive the statements there from only the formal duality principal. On the other hand, here are some results we did not present before.

Proposition 6.3 *Consider the following pushout square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"', "\simeq"] \arrow[dr, phantom, "\lrcorner" very near end] & B \arrow[d, "k"] \\ X \arrow[r, "f'"'] & Y. \end{tikzcd} \tag{6.8}$$

Suppose that j is both a closed cofibration and a homotopy equivalence. Then so is k.

Proof It suffices to show that k has the LLP with respect to all fibrations. So assume $p : E \to B$ is a fibration and maps g, h are given so as to make the following diagram commute

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[d, "j"', "\simeq"] \arrow[dr, phantom, "\lrcorner" very near end] & B \arrow[r, "g"'] \arrow[d, "k"] & E \arrow[d, "p"] \\ X \arrow[r, "f'"'] & Y \arrow[r, "h"'] \arrow[ur, dashed] & B. \end{tikzcd} \tag{6.9}$$

We must construct the dotted arrow. Now, since j is both a cofibration and a homotopy equivalence, there exists a map $\alpha : X \to E$ such that $p\alpha = hf'$ and $\alpha j = gf$. Since the left-hand square is a pushout, there is a unique map $\beta : Y \to E$ with $\beta f' = \alpha$ and $\beta k = g$. Also, $(p\beta)k = pg = hk$ and $(p\beta)f' = p\alpha = hf'$, and this implies that $p\beta = h$, so we're done. ■

This proposition is formally dual to the statement of Corollary 4.6. The proof here followed comparatively quickly having established the Orthogonality Theorem.

Proposition 6.4 *Assume given a countable sequence of cofibrations*

$$X_0 \xrightarrow{j_0} X_1 \xrightarrow{j_1} \ldots \to X_n \xrightarrow{j_n} X_{n+1} \to \ldots \tag{6.10}$$

and let $X_\infty = colim\ X_n$. Then the canonical map $X_1 \to X_\infty$ is a cofibration.

Similarly, if

$$\ldots \to E_{n+1} \xrightarrow{p_n} E_n \to \ldots \xrightarrow{p_1} E_1 \xrightarrow{p_0} E_0 \tag{6.11}$$

is a countable sequence of fibrations $p_n : E_n \to E_{n-1}$ and $E_\infty = lim\ E_n$, then the canonical map $\pi : E_\infty \to E_0$ is a fibration.

Proof The first statement is left as an exercise for the reader. To show that π is a fibration it will suffice to show that it has the right lifting property with respect to any given map $j : A \hookrightarrow X$ which is both a closed cofibration and a homotopy equivalence. So suppose given the solid part of the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"', "\simeq"] & E_\infty \arrow[d, "\pi"] \\ X \arrow[r, "g"'] \arrow[ur, dashed, "h"] & E_0 \end{tikzcd} \tag{6.12}$$

We produce the dotted filler h as follows. For each $n \geq 1$ let $f_n : A \to E_n$ and $g_n : X \to E_n$ be the composites

$$f_n : A \to E_\infty \to E_n, \qquad g_n : X \to E_\infty \to E_n. \tag{6.13}$$

Then for each $n \geq 1$ the solid part of

$$\begin{tikzcd} A \arrow[r, "f_n"] \arrow[d, "j"', "\simeq"] & E_n \arrow[d, "p_n"] \\ X \arrow[r, "g_{n-1}"'] \arrow[ur, dashed, "h_n"] & E_{n-1} \end{tikzcd} \tag{6.14}$$

commutes. Since j has the LLP with respect to p_n, the dotted arrow $h_n : X \to E_n$ can be filled in. Set $h_0 = g$. Then the collection of maps h_n, $n \geq 0$, induces a map $h : X \to E_\infty$ into the limit of the tower 6.11. By definition $\pi h = g$. Since each composite

$$A \xrightarrow{j} X \xrightarrow{h} E_\infty \to E_n \tag{6.15}$$

coincides with f_n, we conclude from the universal property of the limit that $hj = f$. In particular h solves the original problem (6.12). ∎

References

[1] F. Borceux, *Handbook of Categorical Algebra I: Basic Category Theory*, Cambridge University Press, 1994.

[2] T. tom Dieck, *Algebraic Topology*, European Mathematical Society, (2008).

[3] A. Strøm, *The homotopy Category is a Homotopy Category*, Archiv der Mathematik **23**, (1972).